

# d-wave holographic superconductors with backreaction in external magnetic fields

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**ABSTRACT:** We study the  $d$ -wave holographic superconductors (the  $d$ -wave model proposed in [arXiv:1003.2991[hep-th]]) immersed in constant external magnetic fields by using the analytic matching method and numerical computation. In the probe limit, we calculate the spatially dependent condensate solution in the presence of the magnetism and find that the expression for the upper critical magnetic field satisfies the relation given in the Ginzburg-Landau theory. The result shows that the upper critical field gradually increases to its maximum value  $B_{c2}$  at absolute zero temperature  $T = 0$ , while vanishing at the critical temperature  $T = T_c$ . Moving away from the probe limit, we investigate the effect of spacetime backreaction on the critical temperature and the upper critical magnetic field. The magnetic fields as well as the electric fields acting as gravitational sources reduce the critical temperature of the superconductor and actually result in a dyonic black hole solution to the leading order. We obtain the expression for the upper critical magnetic field up to  $\mathcal{O}(\kappa^2)$  order. The analytic result is consistent with the numerical findings.

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## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. The probe limit</b>	<b>2</b>
2.1 The critical temperature	3
2.2 The upper critical magnetic field	6
<b>3. Backreaction</b>	<b>7</b>
3.1 The critical temperature with backreaction	8
3.2 The upper critical magnetic field with backreaction	12
<b>4. Numerical results</b>	<b>16</b>
<b>5. Conclusions</b>	<b>18</b>

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## 1. Introduction

The gauge/gravity duality [1, 2, 3] has been proved to be a powerful tool for studying the strongly coupled systems in field theory. This duality provides a well-established method for calculating correlation functions in a strongly interacting field theory using a dual classical gravity description. The mechanism of the high temperature superconductors has long been an unsolved mysteries in modern condensed matter physics and the difficulties lie in the strong coupling nature of the theory. Considering the above facts, Gubser first suggested that by coupling the Abelian Higgs model to gravity with a negative cosmological constant, one can find solutions that spontaneously break the Abelian gauge symmetry via a charged complex scalar condensate near the horizon of the black hole[4, 5]. Later, Hartnoll et al proposed a holographic model for  $s$ -wave superconductors by considering a neutral black hole with a charged scalar and the only Maxwell sector  $A = A_t$ . They captured the essence in this limit and showed that the properties of a (2+1)-dimensional superconductor can indeed be reproduced [6]. The gravitational model that dual to the  $d$ -wave superconductors was proposed in [7, 8] where the complex scalar field for the  $s$ -wave model is replaced by a symmetric traceless tensor.

The purpose of this paper is to explore the behavior of the upper critical magnetic field for  $d$ -wave holographic superconductors. We will work in both the probe limit and away from the probe limit. The probe limit corresponds to the case the electric

charge  $q \rightarrow \infty$  or the Newton constant approaches zero. Away from the probe limit at a lower temperature, backreaction on the spacetime is important because the black hole solution becomes hairy and Coulomb energy of the matter field near the black hole horizon becomes larger. The phase diagram thus might be modified. In a more recent paper[9], an analytical calculation on the critical temperature of the Gauss-Bonnet holographic superconductors with backreaction has been presented and confirmed the numerical results that backreaction makes condensation harder[10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. The hairy black hole solution requires to go beyond the probe limit. In [6], it was suggested to take finite  $q$  by setting  $2\kappa^2 = 1$ . Recently, the author in [9] proposed to keep  $2\kappa^2$  finite with setting  $q = 1$  instead. We will follow the latter choice. In [24], it was found analytically that for  $s$ -wave holographic superconductors the presence of the magnetic field results in the depression in  $T_c$ , while the upper value of the critical magnetic field performance is improved (see [25, 26, 27, 28] for related work of analytic study on holographic superconductors). There have been a lot of works focused on the magnetic field effects on the holographic model of condensate matter [29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40]. In this work, we will generalize the analytic discussion on the upper critical magnetic field for  $s$ -wave superconductors in the probe limit [41] to the  $d$ -wave holographic superconductors. We will consider the spacetime backreaction which has not been discussed in the available studies of the critical magnetic field for  $d$ -wave holographic superconductors. Furthermore we will carry out numerical computations to check the analytic results for the critical magnetic field in  $d$ -wave holographic superconductors. We will check whether the result found in [24] is universal which can hold in the  $d$ -wave holographic superconductors.

The organization of the paper is as follows: we first study the condensation of the order parameter in the probe limit in section 2. In the weak field limit, the critical temperature and the order parameter operator will be calculated first. Then we continue the calculation to the strong field limit and obtain an analytical expression for the backreaction on the upper critical magnetic field. In section 3, we study the effect of spacetime backreaction on the critical temperature and the upper critical magnetic field. The presence of the magnetic field actually leads to a dyonic black hole solution and the critical temperature drops due to the backreaction of the magnetic field. In section 4, we show at qualitative level, the analytic study is comparable with the numerical computation. The conclusion will be presented in the last section.

## 2. The probe limit

In this section, we will study the condensate solution and the upper critical magnetic field for  $d$ -wave superconductors by using the matching method. Up to now, two types of  $d$ -wave superconductors have been proposed: One is constructed by using

a symmetric, traceless second-rank tensor field and a  $U(1)$  gauge field in the background of the AdS black hole[7]. The other model for the  $d$ -wave order parameter is dual to a charged massive spin two field propagating in an asymptotically AdS geometry [8]. We will take the first one as the example.

## 2.1 The critical temperature

We adopt the action for the  $d$ -wave superconductor as follows[7]

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ \left( R + \frac{6}{l^2} \right) + \mathcal{L}_m \right\}, \quad (2.1)$$

$$\mathcal{L}_m = -\frac{1}{q^2} \left[ (D_\mu B_{\mu\gamma})^* D^\mu B^{\mu\gamma} + m^2 B_{\mu\gamma}^* B^{\mu\gamma} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] \quad (2.2)$$

where  $B_{\mu\nu}$  is a symmetric traceless tensor,  $D_\mu = \partial_\mu + iA_\mu$  is the covariant derivative,  $q$  and  $m^2$  are the charge and mass squared of  $B_{\mu\nu}$ , respectively. The  $d$ -wave superconductors can condensate on the  $x - y$  plane on the boundary with translational invariance, and the rotational symmetry is broken down to  $Z(2)$  with the condensate changing its sign under a  $\pi/2$  rotation on the  $x - y$  plane. Same as in [7], we use the spatial dependent ansatz for the  $d$ -wave superconductors

$$B_{\mu\nu} = \text{diagonal}(0, 0, \psi(z), -\psi(z)), \quad A = \phi(z)dt. \quad (2.3)$$

In the probe limit, the bulk gravitational theory is described by the AdS-Schwarzschild metric

$$ds^2 = \frac{r^2}{l^2} \left( -f(r)dt^2 + \sum_i^2 dx_i^2 \right) + \frac{l^2}{r^2 f(r)} dr^2, \quad (2.4)$$

where the metric coefficient

$$f(r) = 1 - \frac{Ml^2}{r^3} = 1 - \frac{r_+^3}{r^3}, \quad (2.5)$$

and  $l$  is the AdS radius and  $M$  is the mass of the black hole. The Hawking temperature of the black hole is  $T = \frac{3M^{1/3}}{4\pi l^{4/3}}$ . Setting  $z = \frac{r_+}{r}$ , the metric can be rewritten in the form

$$ds^2 = \frac{l^2 \alpha^2}{z^2} [-f(z)dt^2 + dx^2 + dy^2] + \frac{l^2}{z^2 f(z)} dz^2, \quad (2.6)$$

where

$$f(z) = 1 - z^3, \quad \alpha = \frac{r_+}{l^2} = \frac{4}{3}\pi T. \quad (2.7)$$

The equations of motion of the two field  $\psi(z)$  and  $\phi(z)$  are given by

$$f\psi'' + \left(f' + \frac{2f}{z}\right)\psi' + \frac{2f'}{z}\psi - \frac{4f}{z^2}\psi - \frac{m^2}{z^2}\psi + \frac{\phi^2}{\alpha^2 f}\psi = 0, \quad (2.8)$$

$$\phi'' - \frac{4z^2|\psi|^2}{\alpha^4 f}\phi = 0. \quad (2.9)$$

The equations of motion of the  $d$ -wave superconductors are very similar to the  $s$ -wave model and the matching method should be very efficient. Before solving the above equations, let us impose the boundary condition near the horizon and in the asymptotic AdS region, respectively:

- 1). On the horizon  $z = 1$ , the scalar potential must be vanishing  $\phi = 0$  and  $\psi$  should be regular.
- 2). In the asymptotic AdS region  $z \rightarrow 0$ , the solution of the scalar field behaves like

$$\psi \sim C_{\Delta_-} z^{\Delta_-} + C_{\Delta_+} z^{\Delta_+}, \quad (2.10)$$

where  $\Delta_{\pm} = \frac{-1 \pm \sqrt{17 + 4m^2 l^2}}{2}$ . The coefficients  $C_{\Delta_-}$  represents as the source of the dual operator and  $C_{\Delta_+}$  correspond to the vacuum expectation values of the operator that couples to  $B_{\mu\nu}$  at the boundary theory. We require  $m^2 l^2 \geq -4$  (thus  $\Delta_+ \leq 0$ ) such that the  $C_{\Delta_+}$  term is a constant or vanishing on the boundary. Note that the  $C_{\Delta_-} z^{\Delta_-}$  term does not impose a constraint on  $m^2 l^2$  by requiring that the third term on the left hand side of Eq.(2.8) to be smaller than the other two terms since we have imposed  $C_{\Delta_-} = 0$ . The condensate of the scalar operator  $\mathcal{O}$  in the boundary field theory dual to the field  $B_{\mu\nu}$  is given by

$$\langle \mathcal{O}_{ij} \rangle = \begin{pmatrix} C_{\Delta_+} r_+^{\Delta_+} & 0 \\ 0 & -C_{\Delta_+} r_+^{\Delta_+} \end{pmatrix}. \quad (2.11)$$

In what follows, we choose to set the mass of the  $B_{\mu\nu}$  to be  $m^2 l^2 = -\frac{1}{4}$ , so that  $\Delta_+ = \frac{3}{2}$  and  $\Delta_- = -\frac{5}{2}$ . The asymptotic value of the scalar potential at the boundary has the form

$$\phi(z) = \mu - qz, \quad (2.12)$$

where  $q = \rho/r_+$ . Here  $\mu$  is interpreted as the chemical potential and  $\rho$  as the charge density in the boundary theory.

Now we are going to solve the equations of motion by using the analytic method developed in [42]. Expanding the two field  $\psi$  and  $\phi$  near the horizon

$$\phi(z) = \phi(1) - \phi'(1)(1-z) + \frac{1}{2}\phi''(1)(1-z)^2 + \dots \quad (2.13)$$

$$\psi(z) = \psi(1) - \psi'(1)(1-z) + \frac{1}{2}\psi''(1)(1-z)^2 + \dots \quad (2.14)$$

and noting that regularity at the horizon which gives

$$\psi'(1) = -\frac{23}{12}\psi(1), \quad (2.15)$$

the expression for  $\psi''(1)$  and  $\phi''(1)$  can be derived from (2.8) and (2.9), respectively

$$\psi''(1) = \frac{1897}{288}\psi(1) - \frac{\phi'(1)^2}{18\alpha^2}\psi(1), \quad (2.16)$$

$$\phi''(1) = -\frac{4\psi(1)^2}{3\alpha^4}\phi'(1). \quad (2.17)$$

The approximate solutions for  $\psi$  and  $\phi$  near the horizon can then be written as

$$\phi(z) = -\phi'(1)(1-z) - \frac{2\psi(1)^2}{3\alpha^2}\phi'(1)(1-z)^2, \quad (2.18)$$

$$\psi(z) = \psi(1) + \frac{23}{12}\psi(1)(1-z) + \frac{1}{2}\left(\frac{1897}{288} - \frac{\phi'(1)^2}{18\alpha^2}\right)\psi(1)(1-z)^2. \quad (2.19)$$

Connecting the near horizon solutions (2.18) and (2.19) with the boundary solutions (2.10) and (2.12) at the intermediate point  $z_m = 1/2$  smoothly, we find

$$\mu - \frac{q}{2} = \frac{b}{2} + \frac{a^2b}{6\alpha^2}, \quad (2.20)$$

$$-q = -b - \frac{2a^2b}{3\alpha^2}, \quad (2.21)$$

$$C_{\Delta+}(\frac{1}{2})^{3/2} = \left(\frac{6409}{2304} - \frac{b^2}{144\alpha^2}\right)a, \quad (2.22)$$

$$\frac{3}{2}C_{\Delta+}(\frac{1}{2})^{1/2} = \left(-\frac{3001}{576} + \frac{b^2}{36\alpha^2}\right)a, \quad (2.23)$$

where we have defined  $-\phi'(1) = b$  and  $\psi(1) = a$ . From (2.20) and (2.21), we obtain

$$\mu = \frac{b}{4} + \frac{3}{4}q, \quad (2.24)$$

$$a = \sqrt{\frac{3}{2}}\sqrt{\frac{q}{b}}\alpha^2\sqrt{1 - \frac{b}{q}}. \quad (2.25)$$

In order to evaluate the expectation value of the operator  $\langle \mathcal{O}_{\frac{3}{2}} \rangle = \sqrt{2}C_{\Delta+}r_+^{3/2}$ , we eliminate the  $ab^2$  term from (2.21) and (2.23) and obtain

$$C_{\Delta+} = \frac{71}{\sqrt{42}}\sqrt{2}a. \quad (2.26)$$

For non-vanishing  $a$ , we can eliminate  $C_{\Delta+}$  to obtain

$$b = \frac{\alpha}{4}\sqrt{\frac{31231}{7}}. \quad (2.27)$$

By further using the relation  $q = \frac{\rho}{r_+}$ ,  $\alpha = \frac{4}{3}\pi T$ , the expectation value of the operator  $\langle \mathcal{O}_{\frac{3}{2}} \rangle$  is given by

$$\mathcal{O}_{\frac{3}{2}} = \frac{2272}{567}\sqrt{6}\pi^3T^2l^2T_c\sqrt{1 + \frac{T}{T_c}}\sqrt{1 - \frac{T}{T_c}}, \quad (2.28)$$

where the critical temperature is defined as

$$T_c = \frac{3\sqrt{\rho}}{2\pi(\frac{31231}{7})^{1/4}} \simeq 0.058\sqrt{\rho}. \quad (2.29)$$

The expectation value give in (2.28) shows us that the  $d$ -wave condensate is indeed the second order phase transition with the mean field critical exponent  $1/2$ .

## 2.2 The upper critical magnetic field

Now we consider the case when the  $d$ -wave superconductor is immersed in a strong external magnetic field. To the leading order, we consider the ansatz

$$\psi = \psi(x, y, z), \quad \phi = \phi_0(z), \quad A_y = B_{c2}x. \quad (2.30)$$

The equation of motion for the  $B_{\mu\nu}$  field now becomes

$$\begin{aligned} f\psi'' + \left(f' + \frac{2f}{z}\right)\psi' + \frac{1}{\alpha^2}(\partial_x^2 + \partial_y^2)\psi + \frac{2iA_y}{\alpha^2}\partial_y\psi \\ + \left(\frac{2f'}{z} - \frac{4f}{z^2} - \frac{m^2}{z^2} - \frac{A_y^2}{\alpha^2}\right)\psi + \frac{\phi^2}{\alpha^2 f}\psi = 0, \end{aligned} \quad (2.31)$$

where the prime ' denotes the derivative with respect to  $z$ . By assuming  $\psi = e^{ipy}F(x, z; p)$ , the above equation of motion can be changed into

$$\left[f\partial_z^2 + \left(f' + \frac{2f}{z}\right)\partial_z + \frac{2f'}{z} + \frac{\phi^2}{\alpha^2 f} - \frac{4f}{z^2} - \frac{m^2}{z^2}\right]F(x, z; p) = \frac{1}{\alpha^2}\left[-\partial_x^2 + (p - B_{c2}x)^2\right]F(x, z; p). \quad (2.32)$$

This equation can again be solved by separating  $F$  as  $F(x, z; p) = X_n(x; p)R_n(z)$ . We then obtain the following eigen equations

$$(-\partial_U^2 + \frac{U^2}{4})X_n(x; p) = \frac{\lambda_n}{2}X_n(x; p), \quad (2.33)$$

$$R_n'' + \left(\frac{f'}{f} + \frac{2}{z}\right)R_n' = \left(\frac{m^2}{z^2 f} - \frac{\phi^2}{\alpha^2 f^2} + \frac{4}{z^2} - \frac{2f'}{zf} + \frac{B_{c2}\lambda_n}{\alpha^2 f}\right)R_n(z), \quad (2.34)$$

where  $U = \sqrt{2B_{c2}}(x - \frac{p}{B_{c2}})$ . Eq.(2.33) can be solved by the Hermite polynomials as follows

$$X_n(x; p) = e^{-U^2/4}H_n(x), \quad (2.35)$$

where  $\lambda_n = 2n + 1$  is the corresponding eigenvalue and  $n = 0, 1, 2, \dots$  denotes the Landau energy level. We will focus on  $n = 0$  case, which corresponds to the droplet solution. Now we solve the equation (2.34) in the strong field limit. Firstly, let us expand  $R(z)$  near the horizon

$$R_0(z) = R_0(1) - R_0'(1)(1 - z) + \frac{1}{2}R_0''(1)(1 - z)^2 + \dots \quad (2.36)$$

Note that the regularity at the horizon gives

$$R_0'(1) = -\frac{23}{12}R_0(1) - \frac{B_{c2}}{3\alpha^2}R_0(1). \quad (2.37)$$

On the other hand, near the AdS boundary  $z \rightarrow 0$ , it sets

$$R_0(z) = C_{\Delta+}z^{\frac{3}{2}}. \quad (2.38)$$

From (2.34), the second order coefficients of  $R_0(z)$  can be calculated as

$$R_0''(1) = \frac{1897}{288}R_0(1) + \frac{47B_{c2}}{36\alpha^2}R_0(1) + \frac{B_{c2}^2}{18\alpha^4}R_0(1) - \frac{\phi'(1)^2}{18\alpha^2}R_0(1) \quad (2.39)$$

Finally, we find the approximate solution near the horizon

$$R_0(z) = R_0(1) + \left(\frac{23}{12} + \frac{B_{c2}}{3\alpha^2}\right)R_0(1)(1-z) + \frac{1}{2}\left(\frac{1897}{288} + \frac{B_{c2}^2}{18\alpha^4} + \frac{47B_{c2}}{36\alpha^2} - \frac{\phi'(1)^2}{18\alpha^2}\right)R_0(1)(1-z)^2 \quad (2.40)$$

Now let us match the solutions (2.38) and (2.40) at the intermediate point  $z_m = 1/2$ . Requiring the solutions to be connected smoothly, we have

$$C_{\Delta+}(\frac{1}{2})^{3/2} = \left(\frac{6409}{2304} + \frac{B_{c2}^2}{144\alpha^4} - \frac{b^2}{144\alpha^2} + \frac{95B_{c2}}{288\alpha^2}\right)R_0(1), \quad (2.41)$$

$$\frac{3}{2}C_{\Delta+}(\frac{1}{2})^{1/2} = \left(-\frac{3001}{576} - \frac{B_{c2}^2}{36\alpha^4} + \frac{b^2}{36\alpha^2} - \frac{71B_{c2}}{72\alpha^2}\right)R_0(1). \quad (2.42)$$

From the above equations, we find the solution for  $b^2 = |\phi'(1)^2|$

$$|\phi'(1)^2| = \frac{569}{14}B_{c2} + \frac{B_{c2}^2}{\alpha^2} + \frac{31231}{112}\alpha^2. \quad (2.43)$$

When the external magnetic field  $B_{c2}$  is vanishing, the above result return to (2.27), which is crucial for the expression of the critical temperature given in (2.29). By plugging  $|\phi'(1)| = \frac{3\rho}{4\pi T}$ ,  $\alpha = \frac{4\pi T}{3}$  and (2.29) into (2.43), we obtain the ansatz for  $B_{c2}$

$$B_{c2} = \frac{4\pi^2}{63}T_c^2 \left( \sqrt{105144\frac{T^4}{T_c^4} + 218617} - 569\frac{T^2}{T_c^2} \right) \quad (2.44)$$

Figure 1 shows the temperature dependence of the upper critical magnetic field, which is consistent with the Ginzburg-Landau theory. We know that for type II superconductors, in the upper critical field the cores of the vortices are nearly touching and the flux contained in each core is only one quantum flux, the average magnetic field is then  $B_{c2} \sim \frac{\Phi_0}{\pi\xi^2(T)}$ . In this sense, we can determine the superconducting coherence length

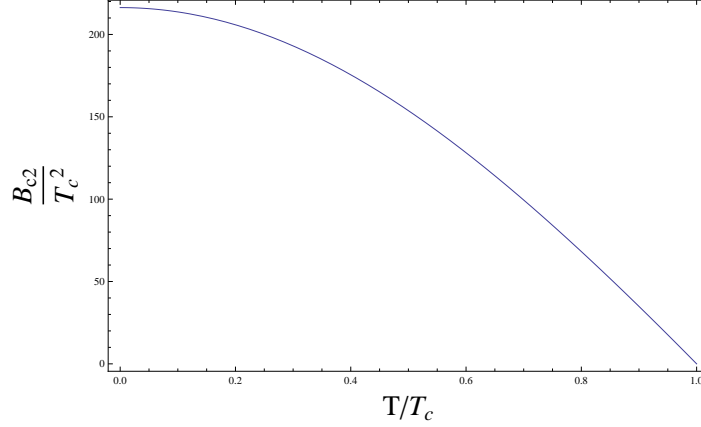
$$\xi(T) \propto (1 - \frac{T}{T_c})^{-1/2}, \quad (2.45)$$

where the critical exponent  $-1/2$  is what wanted for  $\xi(T)$  in the Ginzburg-Landau theory.

### 3. Backreaction

Away from the probe limit, the background spacetime becomes non-neutral and hairy. Both the electric field and the magnetic field could backreact on the background geometry. In this section, we first consider the backreaction of the electric field and then we turn to discuss the backreaction of the external magnetic field.





**Figure 1:** (color online) The upper critical magnetic field changes with  $T/T_c$  for  $d$ -wave superconductors.  $T_c$  denotes the critical temperature without external magnetic field.

### 3.1 The critical temperature with backreaction

The hairy black hole solution is assumed to take the following metric ansatz

$$ds^2 = -f(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{l^2}(dx^2 + dy^2), \quad (3.1)$$

The  $tt$  and  $rr$  components of the background Einstein equation yield

$$f' + \frac{f}{r} - \frac{3r}{l^2} + \kappa^2 r \left[ \frac{e^\chi}{2} \phi'^2 + m^2 \psi^2 + f \left( \frac{\psi^2 \phi^2 e^\chi}{f^2} + \psi'^2 \right) \right] = 0, \quad (3.2)$$

$$\chi' + 2\kappa^2 r (\psi'^2 + \frac{\psi^2 \phi^2 e^\chi}{f^2}) = 0. \quad (3.3)$$

When the Hawking temperature is above a critical temperature  $T > T_c$ , the solution is the well-known AdS-Reissner-Nordström black holes

$$f = \frac{r^2}{l^2} - \frac{1}{r} \left( \frac{r_+^3}{l^2} + \frac{\kappa^2 \rho^2}{2r_+} \right) + \frac{\kappa^2 \rho^2}{2r^2}, \quad \chi = \psi = 0, \quad \phi = \rho \left( \frac{1}{r_+} - \frac{1}{r} \right). \quad (3.4)$$

Near the critical temperature  $T \sim T_c$ , the AdS-Reissner-Nordström solution becomes unstable against perturbation of the scalar field because the coupling of the scalar to gauge field induces an effective negative mass term for the scalar field. This negative mass term will drive the scalar field tachyonic as the temperature is lowered at fixed charge density. At the asymptotic AdS boundary ( $r \rightarrow \infty$ ), the scalar and the Maxwell fields behave as

$$\psi = \frac{\langle \mathcal{O}_{\Delta_-} \rangle}{r^{\Delta_-}} + \frac{\langle \mathcal{O}_{\Delta_+} \rangle}{r^{\Delta_+}}, \quad \phi = \mu - \frac{\rho}{r} + \dots \quad (3.5)$$

where  $\mu$  and  $\rho$  are interpreted as the chemical potential and charge density of the dual field theory on the boundary.

The equations of motion for field  $B_\mu$  and corresponding Maxwell field  $\phi$  are

$$\psi'' + \left(\frac{f'}{f} - \frac{\chi'}{2} - \frac{2}{r}\right)\psi' + \left(\frac{\phi^2 e^\chi}{f^2} - \frac{m^2}{f} - \frac{2f'}{rf}\right)\psi = 0, \quad (3.6)$$

$$\phi'' + \left(\frac{2}{r} + \frac{\chi'}{2}\right)\phi' - \frac{4\psi^2}{r^4 f}\phi = 0. \quad (3.7)$$

Near the critical temperature, the order parameter is small-valued and one can consider it as an expansion parameter

$$\epsilon \equiv \langle \mathcal{O}_{\Delta_+} \rangle. \quad (3.8)$$

It is worth noting that given the structure of our equations of motion, only the even orders of  $\epsilon$  in the gauge field and gravitational field, and odd orders of  $\epsilon$  in the scalar field appear here. So, we expand the scalar field  $\psi$ , the gauge field as series in  $\epsilon$

$$\phi = \phi_0 + \epsilon^2 \phi_2 + \epsilon^4 \phi_4 + \dots \quad (3.9)$$

$$\psi = \epsilon \psi_1 + \epsilon^3 \psi_3 + \epsilon^5 \psi_5 + \dots \quad (3.10)$$

The background metric elements  $f(z)$  and  $\chi(z)$  can be expanded around the AdS-Reissner-Nordström solution

$$f = f_0 + \epsilon^2 f_2 + \epsilon^4 f_4 + \dots \quad (3.11)$$

$$\chi = \epsilon^2 \chi_2 + \epsilon^4 \chi_4 + \dots \quad (3.12)$$

The chemical potential  $\mu$  should also be corrected order by order

$$\mu = \mu_0 + \epsilon^2 \delta\mu_2, \quad (3.13)$$

where  $\delta\mu_2$  is positive. Therefore, near the phase transition, the order parameter as a function of the chemical potential, has the form

$$\epsilon = \left( \frac{\mu - \mu_0}{\delta\mu_2} \right)^{1/2}. \quad (3.14)$$

We can see that when  $\mu$  approaches  $\mu_0$ , the order parameter becomes zero and phase transition happens. So the critical value of  $\mu$  is  $\mu_c = \mu_0$ . The critical exponent  $1/2$  is the universal result from the Ginzburg-Landau mean field theory. The equation for  $\phi$  is solved at zeroth order by  $\phi_0 = \mu_0(1 - z)$  and this gives a relation  $\rho = \mu_0 r_+$ . Thus, to zeroth order the equation for  $f$  is solved as

$$f_0(z) = \frac{r_+^2}{z^2 l^2} \left( 1 - z \right) \left( 1 + z + z^2 - \frac{\kappa^2 l^2 \mu_0^2}{2r_+^2} z^3 \right). \quad (3.15)$$

Now the horizon locates at  $z = 1$ . We will see that the critical temperature with spacetime backreaction can be determined by solving the equation of motion for  $\psi$  to the first order. Using matching method at the horizon, we obtain

$$\psi'_1(1) = \left(\frac{r_+^2 m^2}{f_0^2(1)} - 2\right)\psi_1(1). \quad (3.16)$$

At the asymptotic AdS boundary, we have

$$\psi_1 = C_+ z^{\Delta_+}. \quad (3.17)$$

We expand  $\psi_1$  in a Taylor series

$$\psi_1 = \psi_1(1) - \psi'_1(1)(1 - z) + \frac{1}{2}\psi''_1(1)(1 - z)^2. \quad (3.18)$$

The second order of  $\psi''_1$  can be obtained by the equation(3.6)

$$\psi''_1(1) = -\left(5 + \frac{f''_0(1)}{2f'_0(1)} - \frac{r_+^2 m^2}{2f'_0(1)^2}\right)\psi'_1(1).$$

Then the expression of  $\psi_1$  can be rewritten as

$$\begin{aligned} \psi_1(z) = & \psi_1(1) - \left(2 + \frac{r_+^2 m^2}{f_0^2(1)}\right)\psi_1(1)(1 - z) - \frac{1}{2}\left[\frac{f''_0(1)}{f'_0(1)} + \frac{r_+^2 \phi'^2}{2f'_0(1)^2}\right. \\ & \left.+ 3 + \left(5 + \frac{f''_0(1)}{2f'_0(1)} - \frac{r_+^2 m^2}{2f'_0(1)^2}\right)\left(2 + \frac{r_+^2 m^2}{f_0^2(1)}\right)\right]\psi_1(1)(1 - z)^2. \end{aligned} \quad (3.19)$$

Since the field function and its first derivative should be connected smoothly, we can properly match the expressions of field function at a middle point

$$\begin{aligned} z_m^{\Delta_+} C_+ = & \psi_1(1) - \left(2 + \frac{r_+^2 m^2}{f_0^2(1)}\right)\psi_1(1)(1 - z_m) - \frac{1}{2}\left[3 + \frac{f''_0(1)}{f'_0(1)} + \frac{r_+^2 \phi'^2}{2f'_0(1)^2}\right. \\ & \left.+ \left(5 + \frac{f''_0(1)}{2f'_0(1)} - \frac{r_+^2 m^2}{2f'_0(1)^2}\right)\left(2 + \frac{r_+^2 m^2}{f_0^2(1)}\right)\right]\psi_1(1)(1 - z_m)^2, \end{aligned} \quad (3.20)$$

$$\begin{aligned} \Delta_+ z_m^{\Delta_+ - 1} C_+ = & \left(2 + \frac{r_+^2 m^2}{f_0^2(1)}\right)\psi_1(1)(1 - z_m) + \left[3 + \frac{f''_0(1)}{f'_0(1)} + \frac{r_+^2 \phi'^2}{2f'_0(1)^2}\right. \\ & \left.+ \left(5 + \frac{f''_0(1)}{2f'_0(1)} - \frac{r_+^2 m^2}{2f'_0(1)^2}\right)\left(2 + \frac{r_+^2 m^2}{f_0^2(1)}\right)\right]\psi_1(1)(1 - z_m). \end{aligned} \quad (3.21)$$

Solving  $C_+$  from the above two equations, we have

$$C_+ = \frac{z_m^{1-\Delta_+} (-4f'_0(1) + m^2 r_+^2 + 2f'_0(1)z_m - m^2 r_+^2 z_m)}{f'_0(1)(-2z_m - \Delta_+ + z_m \Delta_+)}, \quad (3.22)$$

Plugging the expression (3.22) into the equation(3.20), we get

$$0 = \frac{m^4 r_+^4 (z_m - 1)}{2f'_0(1)^2} + \frac{f''_0(1)}{f'_0(1)^2} \left( \frac{m^2 r_+^2}{2} - \frac{1}{2} m^2 r_+^2 z_m \right) + \frac{4\Delta_+ - 2z_m \Delta_+}{-2z_m - \Delta_+ + z_m \Delta_+} \\ + \frac{m^2 r_+^2}{f'_0(1)} \left( 7 - 6z_m + \frac{(z_m - 1)\Delta_+}{-2z_m - \Delta_+ + z_m \Delta_+} \right) + \frac{r_+^2 \phi^2(1 - z_m)}{2f'_0(1)^2} - 9 + 7z_m \quad (3.23)$$

After plugging  $f'_0(1) = -\frac{3r_+^2}{l^2} + \frac{\kappa^2 \mu_0^2}{2}$ ,  $f''_0(1) = \frac{6r_+^2}{l^2} + \kappa^2 \mu_0^2$  and  $\phi'_0(1) = -\mu_0$  into equation(3.23), we obtain

$$0 = -\frac{r_+^4(162 - 126z_m)}{l^4} - \frac{m^2 r_+^4(36 - 30z_m)}{l^2} + \frac{r_+^4 \Delta_+(72 - 36z_m)}{l^4(-2z_m - \Delta_+ + z_m \Delta_+)} \\ + m^4 r_+^4 (z_m - 1) + \frac{6m^2 r_+^4 \Delta_+(1 - z_m)}{l^2(-2z_m - \Delta_+ + z_m \Delta_+)} + \left[ r_+^2(1 - z_m) \right. \\ \left. + \frac{\kappa^2 r_+^2(54 - 42z_m)}{l^2} + \frac{12\kappa^2 r_+^2 \Delta_+(z_m - 2)}{l^2(-2z_m - \Delta_+ + z_m \Delta_+)} + \frac{\kappa^2 m^2 r_+^2 \Delta_+(z_m - 1)}{-2z_m - \Delta_+ + z_m \Delta_+} \right. \\ \left. + \kappa^2 m^2 r_+^2(8 - 7z_m) \right] \mu_0^2 + \left[ \frac{(7z_m - 9)}{2} + \frac{\Delta_+(2 - z_m)}{-2z_m - \Delta_+ + z_m \Delta_+} \right] \kappa^4 \mu_0^4, \quad (3.24)$$

Neglecting the  $\kappa^4$  terms, we get

$$\mu_0 = \frac{r_+}{l^2 \sqrt{1 - z_m}} \left[ (162 - 126z_m) + m^2 l^2 (36 - 30z_m) - \frac{\Delta_+(72 - 36z_m)}{-2z_m - \Delta_+ + z_m \Delta_+} \right. \\ \left. + m^4 l^4 (1 - z_m) - \frac{6m^2 l^2 \Delta_+(1 - z_m)}{-2z_m - \Delta_+ + z_m \Delta_+} \right]^{\frac{1}{2}} \left\{ 1 + \kappa^2 \left[ \frac{(54 - 42z_m)}{l^2(1 - z_m)} \right. \right. \\ \left. \left. + \frac{12\Delta_+(z_m - 2)}{l^2(-2z_m - \Delta_+ + z_m \Delta_+)(1 - z_m)} - \frac{m^2 l^2 \Delta_+}{-2z_m - \Delta_+ + z_m \Delta_+} \right. \right. \\ \left. \left. + \frac{m^2 l^2(8 - 7z_m)}{(1 - z_m)} \right] \right\}^{-\frac{1}{2}}. \quad (3.25)$$

Further using the relation  $\mu_0 = \frac{\rho}{r_+}$ , we obtain

$$r_+ = \sqrt{\rho} l^2 \sqrt{1 - z_m} \left[ (162 - 126z_m) + m^2 l^2 (36 - 30z_m) - \frac{\Delta_+(72 - 36z_m)}{-2z_m - \Delta_+ + z_m \Delta_+} \right. \\ \left. + m^4 l^4 (1 - z_m) - \frac{6m^2 l^2 \Delta_+(1 - z_m)}{-2z_m - \Delta_+ + z_m \Delta_+} \right]^{-\frac{1}{2}} \left\{ 1 + \kappa^2 \left[ \frac{(54 - 42z_m)}{l^2(1 - z_m)} \right. \right. \\ \left. \left. + \frac{12\Delta_+(z_m - 2)}{l^2(-2z_m - \Delta_+ + z_m \Delta_+)(1 - z_m)} - \frac{m^2 l^2 \Delta_+}{-2z_m - \Delta_+ + z_m \Delta_+} \right. \right. \\ \left. \left. + \frac{m^2 l^2(8 - 7z_m)}{(1 - z_m)} \right] \right\}^{\frac{1}{2}}. \quad (3.26)$$

The Hawking temperature in this model is

$$T = \frac{3r_+}{4\pi l^2} \left( 1 - \frac{\kappa^2 l^2 \mu_0^2}{6r_+^2} \right), \quad (3.27)$$

At the critical point, there exists a relationship  $T = T_c$ . Substituting  $\mu_0$  and  $r_+$  we can obtain

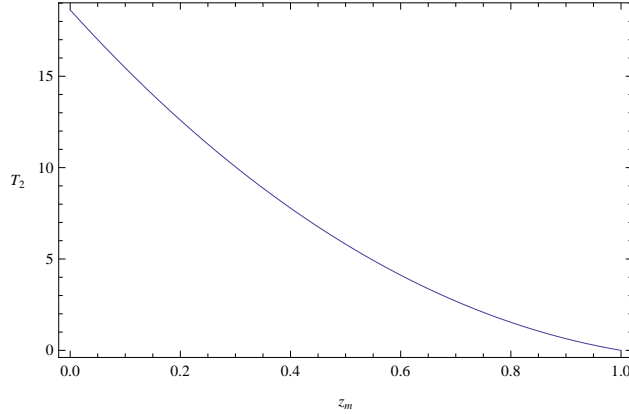
$$T_c = T_1(1 - \frac{2\kappa^2}{l^2}T_2), \quad (3.28)$$

where

$$T_1 = \frac{3\sqrt{\rho}}{4\pi}\sqrt{1-z_m}\left[(162-126z_m)+m^2l^2(36-30z_m)-\frac{\Delta_+(72-36z_m)}{-2z_m-\Delta_++z_m\Delta_+}+m^4l^4(1-z_m)-\frac{6m^2l^2\Delta_+(1-z_m)}{-2z_m-\Delta_++z_m\Delta_+}\right]^{-\frac{1}{2}}, \quad (3.29)$$

$$T_2 = \frac{1}{2l^2(1-z_m)\sqrt{1-z_m}}\left[(162-126z_m)+m^2l^2(36-30z_m)-\frac{\Delta_+(72-36z_m)}{-2z_m-\Delta_++z_m\Delta_+}+m^4l^4(1-z_m)-\frac{6m^2l^2\Delta_+(1-z_m)}{-2z_m-\Delta_++z_m\Delta_+}\right]. \quad (3.30)$$

When setting  $\Delta_+ = \frac{3}{2}$ ,  $z_m = \frac{1}{2}$  and  $m^2l^2 = -\frac{1}{4}$ , we have  $T_c = 0.058\sqrt{\rho}(1 - 5.809\frac{\kappa^2}{l^2})$ , which agrees with the result obtained in (2.29). Figure 2 shows that  $T_2$  is positive in the range  $0 \leq z_m \leq 1$ . This means that the effects of the backreaction can make the condensation harder to be formed. This result also agrees with the results obtained in [9, 10, 11, 18, 24].



**Figure 2:**  $T_2$  is positive for an arbitrary value of  $z_m$ . Note that  $\Delta_+ = \frac{3}{2}$  and  $m^2l^2 = -\frac{1}{4}$ .

### 3.2 The upper critical magnetic field with backreaction

In this part, we study the behavior of a d-wave holographic superconductor which is exposed to an external magnetic field and simultaneously consider the backreaction. As pointed out in [41], one may regard the scalar field  $\psi$  as a perturbation and examine its behavior in the neighborhood of the upper critical magnetic field  $B_{c2}$ . In this case,  $\psi$  is a function of the bulk coordinate  $z$  and the boundary coordinates  $(x, y)$  simultaneously. According to the AdS/CFT correspondence, if the scalar field

$\psi \sim X(x, y)R(z)$ , the vacuum expectation values  $\langle \mathcal{O} \rangle \propto X(x, y)R(z)$  at the asymptotic AdS boundary (i.e.  $z \rightarrow 0$ ) [33, 43]. One can simply write  $\langle \mathcal{O} \rangle \propto R(z)$  by dropping the overall factor  $X(x, y)$ . To the leading order, we have

$$\psi_1 = \psi_1(x, y, z), \quad A_t = \phi_0(z), \quad A_x = 0, \quad A_y = B_{c2}x. \quad (3.31)$$

One can check that to the leading order, the ansatz given above is proper for the following calculation. Moreover, the black hole carries both electric and magnetic charge and the bulk Maxwell field implies that

$$A = B_{c2}x dy + \phi_0 dt. \quad (3.32)$$

The constant external magnetic field may also backreact on the background black hole geometry. The  $tt$  and  $rr$  components of the Einstein's equations in the presence of the magnetic field at the leading order become

$$\begin{aligned} f'_0 + \frac{f_0}{r} - \frac{3r}{l^2} + \kappa^2 r \left[ \frac{e^x}{2} \phi'^2 + m^2 \psi^2 + f_0 \left( \psi'^2 + \frac{q^2 \phi^2 \psi^2 e^x}{f_0^2} \right) + \frac{B_{c2}^2 l^4}{4r^2} \right] &= 0, \\ \chi' + 2\kappa^2 r \left( \psi'^2 + \frac{q^2 \phi^2 \psi^2 e^x}{f_0^2} \right) &= 0. \end{aligned} \quad (3.33)$$

Note that near the critical temperature, we have expanded  $f(z)$  and  $\chi(z)$  as

$$f = f_0 + \epsilon^2 f_2 + \epsilon^4 f_4 + \dots \quad (3.34)$$

$$\chi = \epsilon^2 \chi_2 + \epsilon^4 \chi_4 + \dots \quad (3.35)$$

It is worth noting that the background metric is the dyonic black hole in  $AdS_4$  [44] because of the magnetic field. To the zeroth order,  $f$  is solved as

$$f_0 = \frac{r_+^2}{z^2 l^2} (1 - z) \left( 1 + z + z^2 - \frac{\kappa^2 l^2 \mu_0^2 z^3}{2r_+^2} - \frac{\kappa^2 l^4 B_{c2}^2 z^3}{2r_+^4} \right). \quad (3.36)$$

and the Maxwell fields are given by

$$\phi_0 = \mu_0 - \frac{\rho}{r_+} z, \quad A_y = B_{c2}x. \quad (3.37)$$

For more than second order  $\mathcal{O}(\epsilon^2)$ , the spacetime metric and the matter fields should depend on the spatial coordinates  $(x, y)$  and the equations of motion become nonlinear partial differential equations. We restrict ourself to the first order and indeed this is enough to determine the critical temperature and the upper critical magnetic field. Similarly, we have expanded  $\psi$  and  $\phi$  as series in  $\epsilon$

$$\phi = \phi_0 + \epsilon^2 \phi_2 + \epsilon^4 \phi_4 + \dots \quad (3.38)$$

$$\psi = \epsilon \psi_1 + \epsilon^3 \psi_3 + \epsilon^5 \psi_5 + \dots \quad (3.39)$$

At the first order, the equation of motion is

$$\psi_1'' + \left(\frac{f_0'}{f_0} + \frac{4}{z}\right)\psi_1' + \left(\frac{r_+^2\phi^2}{z^4 f_0^2} - \frac{r_+^2 m^2}{z^4 f_0} + \frac{f_0'}{z f_0}\right)\psi_1 = -\frac{l^2}{z^2 f_0} \left[\partial_x^2 + (\partial_y - i B_{c2} x)^2\right] \psi_1. \quad (3.40)$$

We assume a separable form for  $\psi_1, \psi_1 = e^{ik_y y} X_n(x) R_n(z)$ , the equation can be brought to the form,

$$-X_n''(x) + (k_y - i B_{c2} x)^2 X_n(x) = \lambda_n B_{c2} X_n(x), \quad (3.41)$$

$$R_n'' + \left(\frac{f_0'}{f_0} + \frac{4}{z}\right)R_n' + \left(\frac{r_+^2\phi^2}{z^4 f_0^2} - \frac{r_+^2 m^2}{z^4 f_0} + \frac{2f_0'}{z f_0}\right)R_n = \frac{\lambda_n B_{c2} l^2}{z^2 f_0} R_n, \quad (3.42)$$

where  $\lambda_n$  stands for the Landau energy level of the harmonic oscillator equation. Taking account that the equation (3.41) is solved by Hermite polynomials and has a series roots like  $X_n(x) = e^{-\frac{(k_y - i B_{c2} x)^2}{2 B_{c2}}} H_n(x)$ , so the solution of  $\psi_1$  can be written as,

$$\psi_1 = R_0(z) \sum_j c_j e^{ik_j y} X_0(x). \quad (3.43)$$

We work on the lowest mode  $n = 0$  in what follows, which is the first to condensate and is the most stable solution after condensation. Next, we focus on solving the equation (3.42) by the matching method and study the relationship between the critical temperature and the magnetic field away from the probe limit. Regularity at the horizon requires

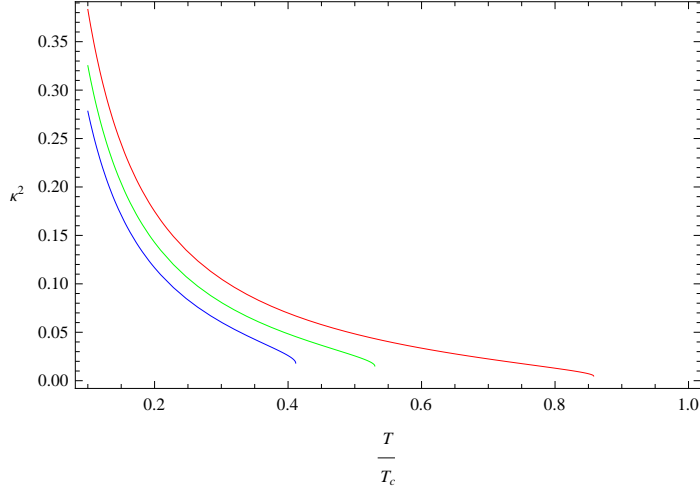
$$R_0'(1) = \frac{r^2 m^2}{f_0'(1)} - 2 + \frac{B_{c2} l^2}{f_0'(1)}. \quad (3.44)$$

Near the AdS boundary,  $R_0$  can be written as  $R_0(z) = C_+ z^{\Delta_+}$ . We expand the  $R_0$  near the horizon,

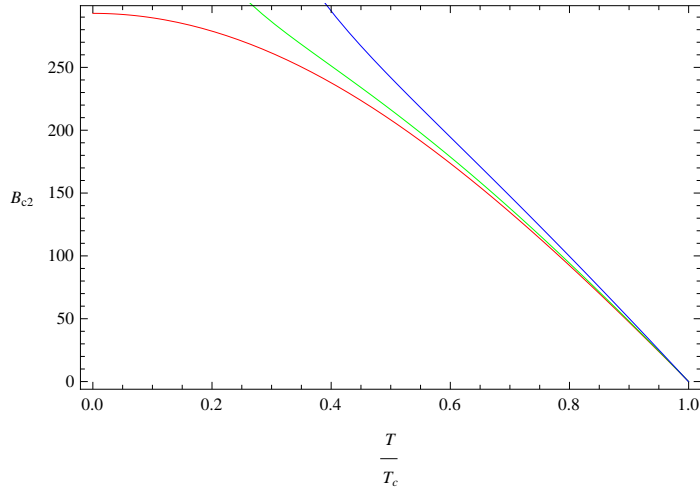
$$R_0(z) = R_0(z)(1) - R_0(z)'(1)(1-z) + \frac{1}{2} R_0(z)''(1)(1-z)^2 + \dots \quad (3.45)$$

Similarly, we can write the expression of  $R_0''(1)$  from (3.42),

$$\begin{aligned} R_0''(1) = & -\frac{1}{2} \left( 5 + \frac{f_0''(1)}{2f_0'(1)} - \frac{B_{c2} l^2}{2f_0'(1)} - \frac{m^2 r^2}{2f_0'(1)} \right) R_0(1)' \\ & + \left( 3 + \frac{f_0''(1)}{f_0'(1)} - \frac{B_{c2} l^2}{f_0'(1)} + \frac{r^2 \phi^2}{2f_0'(1)^2} \right) R_0(1). \end{aligned} \quad (3.46)$$



**Figure 3:** The parameter  $\kappa^2$  shown as a function of  $T/T_c$ . The plot shows analytic result in which Lines from top to bottom are for  $B_{c2} = 100$ ,  $B_{c2} = 1000$ ,  $B_{c2} = 2000$ .



**Figure 4:** The magnetic field  $B_{c2}$  shown as a function of  $T/T_c$ . The plot gives analytic result for  $\kappa^2 = 0$  (red),  $\kappa^2 = 0.0001$  (green),  $\kappa^2 = 0.0004$  (blue) from bottom to top.

Following the above method, we obtain

$$\begin{aligned}
0 = & -B_{c2}^2 + B_{c2}^2 z_m + r_+^4 (-162 - 36m^2 - m^4 + 126z_m + 30m^2 z_m + m^4 z_m) \\
& + r_+^2 (\mu^2 - 30B_{c2} - 2m^2 B_{c2} - \mu^2 z_m + 24B_{c2} z_m + 2m^2 B_{c2} z_m) \\
& + \frac{1}{-\Delta_+ - 2z_m + \Delta_+ z_m} \left[ r_+^2 \Delta_+ B_{c2} (6 - 6z_m) + r_+^4 \Delta_+ (72 + 6m^2 - 36z_m - 6m^2 z_m) \right] \\
& + \kappa^2 \left\{ (54 - 42z_m) B_{c2}^2 + (8 - 7z_m) m^2 B_{c2}^2 + \frac{(7 - 6z_m) B_{c2}^3}{r_+^2} - \frac{(1 - z_m) \Delta_+ B_{c2}^3}{r_+^2 (-\Delta_+ - 2z_m + \Delta_+ z_m)} \right. \\
& - \frac{(24 + m^2 - 12z_m - m^2 z_m) \Delta_+ B_{c2}^2}{-\Delta_+ - 2z_m + \Delta_+ z_m} + \mu^2 \left[ (54 + 8m^2) r_+^2 + (7 - 6z_m) B_{c2} - (42 + 7m^2) r_+^2 z_m \right. \\
& \left. \left. + \frac{(1 - z_m) \Delta_+ B_{c2}}{-\Delta_+ - 2z_m + \Delta_+ z_m} - \frac{(24 + m^2 - 12z_m - m^2 z_m) r_+^2 \Delta_+}{-\Delta_+ - 2z_m + \Delta_+ z_m} \right] \right\}. \tag{3.47}
\end{aligned}$$



The difference between (3.24) and (3.47) comes from the  $B_{c2}$  related terms, from which we can derive the relationship between  $B_{c2}$  and  $r_+$  as follows

$$B_{c2}^2 = \frac{323761r_+^4}{196} - \frac{16853378855\kappa^2 r_+^4}{19208} - \frac{35278}{49}\kappa^2 r_+^2 \mu_0^2. \quad (3.48)$$

We find

$$T_c = 0.058\sqrt{\rho}(1 - 143.463\frac{\kappa^2}{l^2}), \quad (3.49)$$

which means that the critical temperature in external magnetic field is lower than the one without magnetic field. We stress that (3.48) is not enough to determine the relation among the upper critical magnetic field, the system temperature  $T$  and the critical temperature  $T_c$ . Of course, there is an ambiguity in the choice of the matching radius. But, the result turns out to be fairly insensitive to the choice of it. First substitute (3.25) and (3.48) into the Hawking temperature  $T = \frac{r_+}{4\pi l^2} \left( 3 - \frac{\kappa^2 l^2 \mu_0^2}{2r_+^2} - \frac{\kappa^2 l^2 B_{c2}^2}{2r_+^4} \right)$  and we see that the Hawking temperature plays the role of the critical temperature in the presence of magnetic fields. Taking advantage of the relation  $\mu_0 = \frac{\rho}{r_+}$  and equation (3.47), we finally obtain

$$B_c \simeq \frac{4\pi^2 T_c^2}{63} \left[ -569 \frac{T^2}{T_c^2} + \sqrt{218617 + 105144 \frac{T^4}{T_c^4}} \right] + \varsigma(T, T_c), \quad (3.50)$$

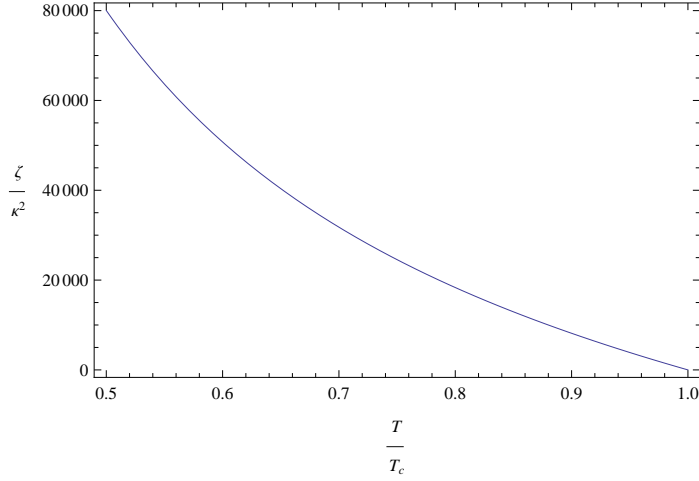
where

$$\varsigma(T, T_c) = 31231\pi^2 \kappa^2 \left[ \frac{31T_c^2}{441T^2} + \frac{105144 \frac{T^4}{T_c^4} + 6949}{5292 \sqrt{218617 + \frac{105144T^4}{T_c^4}}} - \frac{569T^2}{5292T_c^2} \right].$$

The result (3.50) implies that it is only applicable near the critical temperature  $T_c$  because the  $\kappa^2$  term will be divergent in the low temperature limit. One may find that when  $\kappa^2 = 0$ , the result exactly agrees with [41], which is also consistent with the Ginzburg-Landau theory where  $B_{c2} \propto \left( 1 - T/T_c \right)$ . Figure 3 shows that for fixed  $B_{c2}$ , the critical temperature drops as  $\kappa^2$  increases. As shown in Figure 4, the upper critical magnetic field decreases as  $T/T_c$  goes up and vanishes at  $T = T_c$ . We also find that the coefficient of the  $\kappa^2$  term is positive for the system temperature  $T$  (see Fig. 5). This result also agrees with the result obtained in [24].

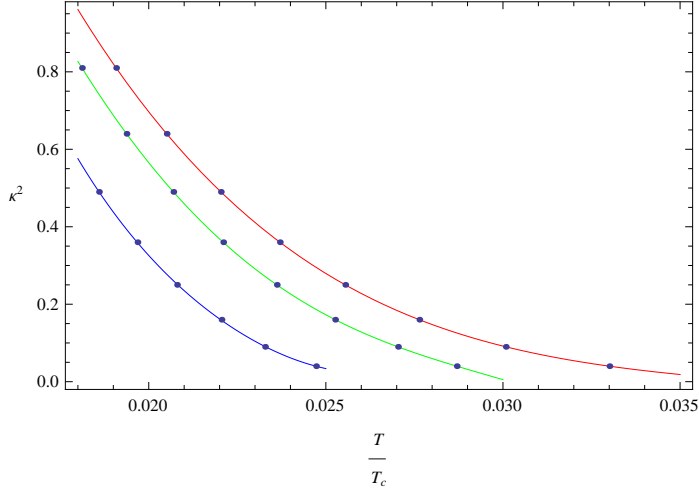
## 4. Numerical results

In this section, we will visually show the relationship among the critical temperature, the upper critical magnetic field and the backreaction parameter  $\kappa^2$  by using the



**Figure 5:** The coefficient of the  $\kappa^2$  term of the upper critical magnetic field as a function of the temperature  $T/T_c$ .

numerical shooting method. We set  $r_+ = 1$  and  $l = 1$  in the numerical computation. Firstly, we want to show how the critical temperature varies as  $\kappa^2$  changes. From Figure 6, one can see that for fixed  $B_{c2}$ , the critical temperature decreases as  $\kappa^2$  rises up. That is to say, when applying the external magnetic field, the backreaction of the spacetime makes condensation harder to happen. This is consistent with the analytic result showed in Figure 3.

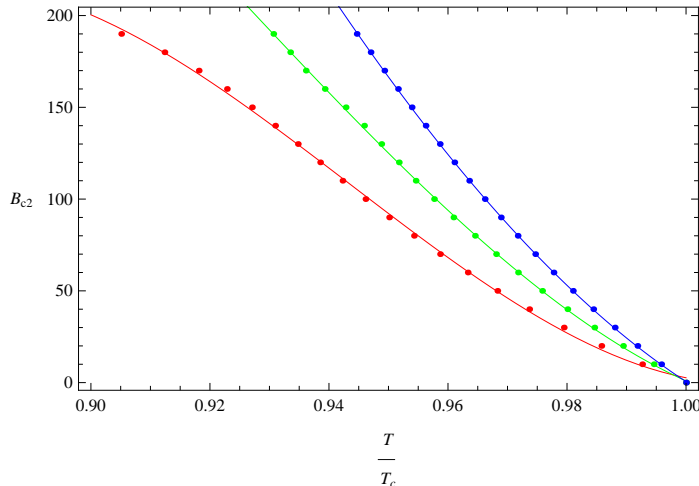


**Figure 6:** The numerical result in which lines from top to bottom are for  $B_{c2} = 100$  (red),  $B_{c2} = 200$  (green),  $B_{c2} = 300$  (blue).

Secondly, we plot the curves between the critical temperature and the upper critical magnetic field  $B_{c2}$  for fixed parameter  $\kappa^2$ . When setting different values of backreaction  $\kappa^2 = 0$ ,  $\kappa^2 = 0.01$ ,  $\kappa^2 = 0.04$  in Figure 7. We find the magnetic field  $B_{c2}$  goes to the opposite direction with increasing  $T/T_c$ . The numerical result

also demonstrates that when  $\kappa^2 = 0$ , the upper critical magnetic field  $B_{c2}$  is finite and smaller than  $\kappa^2 \neq 0$  cases. That is to say, although the critical temperature is significantly suppressed by a non-zero  $\kappa^2$ , the upper bound of  $B_{c2}$  become larger.

The numerical computation is done by using the shooting method. The plot of the analytic result (Figures 3 and 4) depends on the choice of the matching point. But the analytic results given in Figure 3 and 4 are qualitatively in good agreement with the numerical results shown in Figures 6 and 7.



**Figure 7:** The numerical result for Lines from top to bottom are for  $\kappa^2 = 0$  (red),  $\kappa^2 = 0.01$  (green),  $\kappa^2 = 0.04$  (blue) respectively.

## 5. Conclusions

In this work we studied the d-wave holographic superconductors in the probe limit and away from the probe limit by using the analytic and numerical computation. In the probe limit, we obtained analytic expressions for the order parameter, the critical temperature and the upper critical magnetic field. The analytic calculation is useful for gaining insight into the strong interacting system. The result shows that the upper critical field gradually increases to its maximum value  $B_{c2}$  at absolute zero temperature  $T = 0$ , while vanishing at the critical temperature  $T = T_c$ . This behavior satisfies the relation given in the Ginzburg-Landau theory. Away from the probe limit, we found that the backreaction of the electric field results in the AdS-Reissner-Nordström black hole solution, which in turn hinder the formation of the condensation. We obtained the analytic formula for the critical temperature in this case and found  $T_2$  is always positive. Condensation becomes even harder to happen in the presence of a constant external magnetic field. The background spacetime geometry to the leading order becomes a dyonic black hole solution in AdS space. The coefficient of spacetime backreaction on the upper critical magnetic field

is positive, which indicates that the magnetic field becomes strong with respect to the backreaction and is in consistent with the result observed in [24]. We have also shown the corresponding numerical results for each case. Qualitatively our analytic and numerical computational results matched very well.

The method employed in this work is very similar to that used in [41]. But surprisingly, we found that the spacetime backreaction presents us an interesting property of holographic superconductors: While the backreaction causes the suppression of the critical temperature, it can enhance the upper critical magnetic field because the  $\mathcal{O}(\kappa^2)$  term is positive. Moreover, we have checked the analytic result by comparing with numerical computation. In fact, we found that the holographic superconductors with backreaction is very similar to carbon doping in  $\text{MgB}_2$  reported in recent experiments[45]: where with the results of the suppression in  $T_c$ , while the  $B_{c2}$  performance is improved. The consistency of the phenomenon in the external magnetic field in holographic superconductor and  $\text{MgB}_2$  superconductor is interesting and further understanding on their relation is called for.

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